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The Large-Deviation Principle and the BCS Model

Anatoly I. Pilyavsky[1] **and Alexei L. Rebenko**[2]

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The BCS model is investigated by the functional integrals method and Euclidean quantum field theory technique. It permits us to apply some version of the Large Deviation Principle and get the exact solution which was obtained earlier by the approximation Hamiltonian method.

KEY WORDS: Large-deviation principle; BCS model; functional integral.

The Bardeen–Cooper–Schrieffer (BCS) model has been investigated by many authors (e.g., see refs. 1–6). It has served as a test of various methods. Despite the large number of works dealing with these models, interest has not lessened.

We would like to single out the two main methods in the investigation of the BCS model. The first is the approximate Hamiltonian method[1–4] and the second is the functional integral one.[5,6] In this short note we stress that the exact solvability of the BCS model can be proved by using the functional integral method and the large-deviation principle or more exactly usinig the Ellis–Rosen theorem[7] (see ref. 8 for details). We should point out that one can also consider this statement as the mathematically strong proof of the results of Svidzinsky.[6]

The main idea consists in using the Euclidean Fermi-field theory technique to obtain a functional integral representation for the temperature Green functions and canonical partition function. These representations can be written in the form of Gaussian functional integrals:

$$Z_{\Lambda} = Z_0 \int d\mu_{\Lambda}(\Delta)\, d\mu_{\Lambda}(\Delta^*) e^{+|\Lambda| F_{\Lambda}(\Delta, \Delta^*)} \tag{1}$$

[1] Lvov Institute of Trade and Economy.

[2] Institute of Mathematics, Ukrainian Academy of Science, Kiev 252601, Ukraine. e-mail: mathkiev@sovam.com.

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$$G^{(ij)}_{\alpha\beta}(x_1, \tau_1; x_2, \tau_2 | \Lambda) = Z_\Lambda^{-1} \int d\mu_\Lambda(\Delta)\, d\mu_\Lambda(\Delta^*)\, \tilde{G}^{(ij)}_{\alpha\beta}(x_1, \tau_1; x_2, \tau_2 | \Delta, \Delta^*) \times \exp\{|\Lambda| F_\Lambda(\Delta, \Delta^*)\} \tag{2}$$

$$F_\Lambda(\Delta, \Delta^*) = \frac{1}{|\Lambda|} \int_0^\beta d\tau \int_\Lambda dx \int_\Lambda dy\, v(x-y)\, [\Delta(\tau) \tilde{G}^{(11)}_{+-}(x, \tau; y, \tau | \Delta) + \Delta^*(\tau)\, \tilde{G}^{(22)}_{-+}(x, \tau; x, \tau | \Delta)] \tag{3}$$

Here Λ is a finite volume and the Green functions $G^{(ij)}_{\alpha\beta}$, α, $\beta = \pm$, $(i, j) = 1, 2$, are the temperature (normal and anomalous) Green functions of Fermi fields coupled by the external fields $\Delta(\tau)$. They satisfy some finite chain of linear integral equations. This chain can be exactly solved at $\Delta(\tau) = \Delta_0 = \text{const}$. Application of the Ellis–Rosen theorem to (1)–(3) permits us to establish this minimum point Δ_0 which satisfies in turn some nonlinear equation of self-coordination. In the set of the constant functions $\Delta(\tau) = \delta_0 = \text{const}$ the solutions depend on the temperature β. Precisely, there exists $\beta_c \neq 0$ such for $\beta < \beta_c$ the system has the unique trivial solution $\Delta_0 = 0$ and for $\beta > \beta_c$ the solution in general is not unique.

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